

VIBRATION ANALYSIS OF PLATES USING THE MULTIVARIABLE SPLINE ELEMENT METHOD

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Abstract—The vibration analysis of plates using the multivariable spline element method is presented in this paper. The spline functions are applied to construct bending moments, twisting moments and transverse displacement field functions. The spline equations of eigenvalue problems with multiple variables of vibration of plates are derived based on the Hellinger-Reissner mixed variational principle. For simplicity, the boundary conditions which consist of three local spline points are amended to fit any specified boundary conditions. Several numerical solutions of plate vibration analysis are presented which illustrate the accuracy and convergence of the method.

INTRODUCTION

The spline finite element method has been developed in the last 10 years. Antes (1974) and Shih (1979) presented bicubic splines as displacement functions for plate bending problems. Mizusawa *et al.* (1979) used the bicubic spline functions to solve the vibration and buckling problems for skew plates. Shen *et al.* (1987), Shen and Wang (1989) and Shen and Huang (1990) extended the spline element method for analysing static, dynamic vibration and stability of stiffened plate and shell problems. The spline element method with a single variable for analysing plates and shells has fewer unknowns, a high degree of accuracy and can be easily programmed, so it was successfully used to analyse certain types of structures with a regular shape on a microcomputer.

Based on the development of the finite element method, the mixed finite element method which is applied to the plate and the plane stress-strain problems is presented in Herrmann (1967) and Mirza and Olson (1980). A kind of multivariable spline element method for analysing plate bending problems has been presented by Shen and Kan (1991). In the present study, bicubic B splines have been used to construct the bending moments, the twisting moments and the transverse displacement field functions in the analysis of the vibration of plates. The spline element vibration eigenvalue equations are derived based on the Hellinger-Reissner mixed variational principle. To demonstrate the accuracy of this method, several numerical examples are given. The proposed multivariable spline element method is compared with other numerical methods. It is shown to be quite effective for the solution of the vibration of plate problems.

SPLINE INTERPOLATE FUNCTIONS AND FIELD FUNCTIONS

In the multivariable spline finite element method, the bending moments, the twisting moments and the transverse displacement are all chosen as field functions. The field functions of the plate are defined in the form of the Kronecker product of two B cubic spline functions:

$$\{M\} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} \lfloor \Phi(x) \rfloor \oplus \lfloor \Phi(y) \rfloor & & \\ & \circ & \\ \circ & & \lfloor \Phi(x) \rfloor \oplus \lfloor \Phi(y) \rfloor \\ & & & \circ \\ & & & & \lfloor \Phi(x) \rfloor \oplus \lfloor \Phi(y) \rfloor \end{bmatrix} \begin{Bmatrix} \{A\} \\ \{B\} \\ \{C\} \end{Bmatrix}, \quad (1)$$

$$w = \lfloor \Phi(x) \rfloor \oplus \lfloor \Phi(y) \rfloor \{D\}, \quad (2)$$

where

$$[\Phi(x)] = [\phi_{-1}(x)\phi(x)_0(x)_1 \dots \phi(x)_N\phi(x)_{N+1}], \quad (x = x, y) \quad (3)$$

$$\{A\} = [\bar{a}_{-1}\bar{a}_0\bar{a}_1 \dots \bar{a}_N \dots \bar{a}_{N+1}]^T, \quad (4)$$

$$\bar{a}_s = [a_{-1s}a_{0s}a_{1s} \dots a_{Ns} \dots a_{N+1s}], \quad (s = -1, 0, 1, \dots, M, M+1), \quad (5)$$

\oplus = Kronecker product of two matrices, i.e. $[A] \oplus [B] = (a_{ij}[B])$.

The parameter vectors $\{A\}$, $\{B\}$, $\{C\}$ and $\{D\}$ are all constants which are to be determined. Parameter vectors $\{B\}$, $\{C\}$ and $\{D\}$ are similar to parameter vector $\{A\}$. The basis of the splines can be written as follows:

When $N \geq 4$

$$\begin{aligned} \phi_{-1} &= \varphi_3\left(\frac{x}{h} + 1\right), \\ \phi_0(x) &= \varphi_3\left(\frac{x}{h}\right) - 4\varphi_3\left(\frac{x}{h} + 1\right), \\ \phi_1(x) &= \varphi_3\left(\frac{x}{h} - 1\right) - \frac{1}{2}\varphi_3\left(\frac{x}{h}\right) + \varphi_3\left(\frac{x}{h} + 1\right), \\ \phi_2(x) &= \varphi_3\left(\frac{x}{h} - 2\right), \\ &\dots\dots\dots \\ \phi_{N-2}(x) &= \varphi_3\left(\frac{x}{h} - N + 2\right), \\ \phi_{N-1}(x) &= \varphi_3\left(\frac{x}{h} - N + 1\right) - \frac{1}{2}\varphi_3\left(\frac{x}{h} - N\right) + \varphi_3\left(\frac{x}{h} - N - 1\right), \\ \phi_N(x) &= \varphi_3\left(\frac{x}{h} - N\right) - 4\varphi_3\left(\frac{x}{h} - N - 1\right), \\ \phi_{N+1}(x) &= \varphi_3\left(\frac{x}{h} - N - 1\right). \end{aligned} \quad (6)$$

The figures of splines $\varphi_3(x)$, $\varphi_3((x/h) - i)$, $\phi_1(x)$, ($i = -1, 0, 1, \dots, N+1$) are shown in Figs 1, 2 and 3 where,

$$\varphi_3(x) = \frac{1}{6h^3} \begin{cases} (x - x_{i-2})^3 & x_{i-2} \leq x \leq x_{i-1} \\ h^3 + 3h^2(x - x_{i-1}) + 3h(x - x_{i-1})^2 - 3(x - x_{i-1})^3 & x_{i-1} \leq x \leq x_i \\ h^3 + 3h^2(x_{i+1} - x) + 3h(x_{i+1} - x)^2 - 3(x_{i+1} - x)^3 & x_i \leq x \leq x_{i+1} \\ (x_{i+2} - x)^3 & x_{i+1} \leq x \leq x_{i+2} \\ 0 & x_{i+2} < |x| \end{cases} \quad (7)$$

In order to be simplified for the treatment of boundary conditions, for $N \geq 4$, the linear combination of splines has been constructed for the first three splines and the last three splines which are shown in Fig. 3.

At the left-hand point, $x = 0$, we have

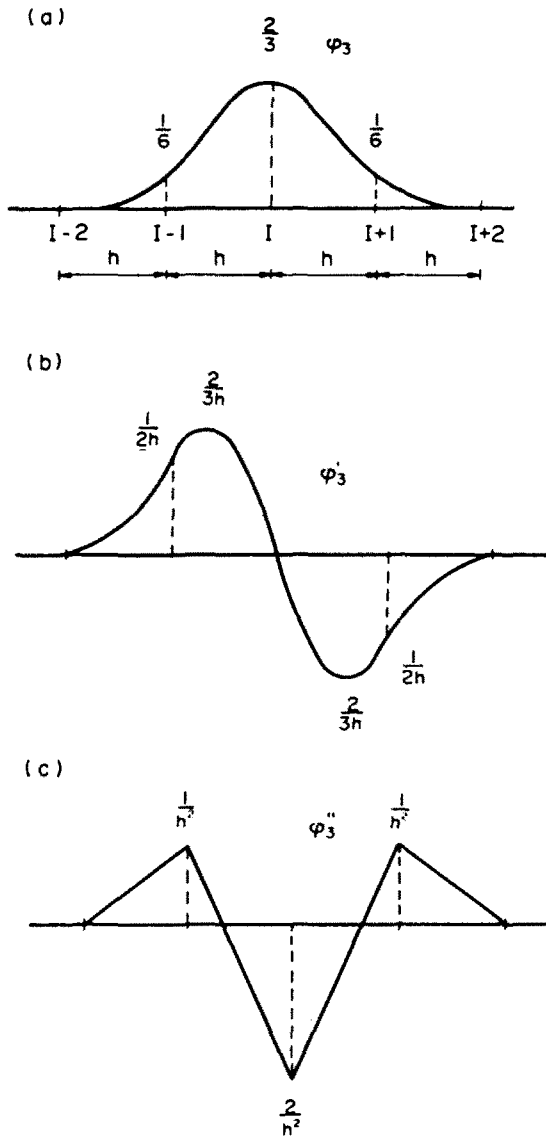


Fig. 1.

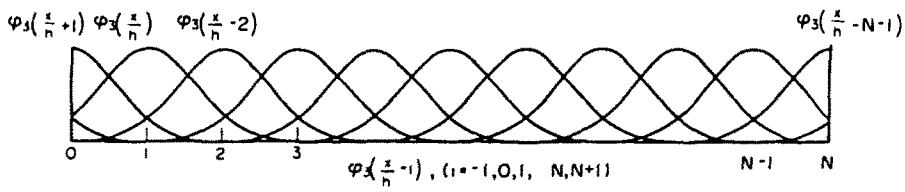


Fig. 2.

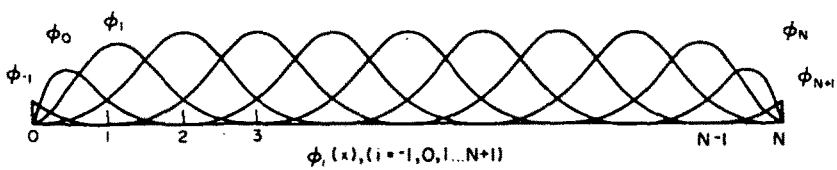


Fig. 3.

$$\phi_i(0) = 0, \quad (i \neq -1),$$

$$\phi'_i(0) = 0, \quad (i \neq -1, 0).$$

At the right-hand point, $x = a$, we have

$$\phi_i(a) = 0, \quad (i \neq N+1),$$

$$\phi'_i(a) = 0, \quad (i \neq N, N+1).$$

MIXED VARIATIONAL PRINCIPLE OF THIN PLATE

The mixed variational principle of thin plates is given by Hu (1981):

$$\begin{aligned} \Pi_2 = & \int_{\Omega} \{M\}^T \{\chi\} dx dy - \int_{\Omega} \frac{1}{2} \{M\}^T [D]^{-1} \{M\} dx dy - \int_{\Omega} qw dx dy \\ & - \int_{c_1+c_2} \left(\frac{\partial M_{ns}}{\partial s} + Qn \right) (w - \bar{w}) ds - \int_{c_1} \bar{q}w ds + \int_{c_1} Mn \left(\frac{\partial w}{\partial n} - \bar{\varphi} \right) ds + \int_{c_2+c_3} Mn \frac{\partial w}{\partial n} ds, \end{aligned} \quad (8)$$

$$c_1 \text{—on fixed edges, } w = \bar{w}, \frac{\partial w}{\partial n} = \bar{\varphi},$$

$$c_2 \text{—on simply-supported edges, } w = \bar{w}, Mn = \bar{M}n,$$

$$c_3 \text{—on free edges, } Mn = \bar{M}n, \frac{\partial Mns}{\partial s} + Qn = \bar{q}.$$

In the case of homogeneous boundary conditions, the mixed functional for vibration problems becomes as follows:

$$\Pi_2 = \int_{\Omega} \int_{\Omega} \{M\}^T \{\chi\} dx dy - \int_{\Omega} \int_{\Omega} \frac{1}{2} \{M\}^T [D]^{-1} \{M\} dx dy - \int_{\Omega} \int_{\Omega} \frac{1}{2} \lambda \bar{m} w^2 dx dy, \quad (9)$$

where

$$\{M\} = [M_x \quad M_y \quad M_{xy}]^T, \quad (10)$$

$$\{\chi\} = [-w_{,xx} - w_{,yy} - 2w_{,xy}]^T \quad (11)$$

$$[D] = \frac{Et^3}{12(1-\mu^2)} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}, \quad (12)$$

\bar{m} —mass density of plate, λ —vibration eigenvalue of plate. If we take the moments and displacements as variations to the mixed functional and make the variation of Π_2 equal to zero, i.e. $\delta\Pi_2 = 0$, we can obtain the equations including the equations of equilibrium, boundary and geometry of the plate.

MULTIVARIABLE SPLINE VIBRATION EIGENMODE EQUATION OF PLATE

The bending moments, twisting moments and transverse displacements used as field variables and the bicubic B_3 spline functions are used to construct the field functions with

multiple variables. Taking eqns (1) and (2) and substituting them into eqn (9) and applying the mixed energy principle, we have

$$\frac{\partial \Pi_2}{\partial \{A\}} = \{0\}, \quad \frac{\partial \Pi_2}{\partial \{B\}} = \{0\}, \quad \frac{\partial \Pi_2}{\partial \{C\}} = \{0\}, \quad \frac{\partial \Pi_2}{\partial \{D\}} = \{0\}, \tag{13}$$

which yields

$$\begin{bmatrix} [F] & [H] \\ [H]^T & [0] \end{bmatrix} \begin{Bmatrix} \{A\} \\ \{B\} \\ \{C\} \\ \{D\} \end{Bmatrix} = \begin{bmatrix} [0] & [0] \\ [0] & \lambda[M] \end{bmatrix} \begin{Bmatrix} \{A\} \\ \{B\} \\ \{C\} \\ \{D\} \end{Bmatrix}, \tag{14}$$

where

$$\begin{bmatrix} -\frac{12}{Et^3} [K_x^{00}] \otimes [K_y^{00}] & \frac{12\mu}{Et^3} [K_x^{00}] \otimes [K_y^{00}] & 0 \\ \text{Symm.} & -\frac{12}{Et^3} [K_x^{00}] \otimes [K_y^{00}] & 0 \\ & & -\frac{24(1+\mu)}{Et^3} [K_x^{00}] \otimes [K_y^{00}] \end{bmatrix}, \tag{15}$$

$$[H] = \begin{bmatrix} -[K_x^{22}] \otimes [K_y^{00}] \\ -[K_x^{00}] \otimes [K_y^{22}] \\ -2[K_x^{01}] \otimes [K_y^{10}] \end{bmatrix}, \tag{16}$$

$$[M] = \tilde{m} [K_x^{00}] \otimes [K_y^{00}], \tag{17}$$

where

$$[K_x^{ij}] = \int_0^a [\Phi^i(x)]^T [\Phi^j(x)] dx, \quad (x = x, y), (i, j = 0, 1, 2). \tag{18}$$

The matrices $[K_x^{ij}]$ ($x = x, y; i, j = 0, 1, 2$) can be found in Shen and Wang (1986). From eqn (14) we have the eigenvalue mode vector equation of the vibration analysis of plates in the following form:

$$([H]^T [F]^{-1} [H] - \lambda [M]) \{D\} = \{0\}. \tag{19}$$

In eqn (19), the kinematic boundary conditions are easily treated just like the finite element method, but they do not relate to the displacements, only to the spline node parameters.

NUMERICAL EXAMPLES

Generally, by using refined meshes of spline knots and higher order splines, we can obtain better approximate solutions using the multivariable spline element method. However, the spline functions with more knots and higher order splines can accelerate the convergence for numerical results, but more variables are involved which leads to excessive CPU time, so that it is not suitable for practical computations. Based on our experience, using the bicubic splines and meshes of 4×4 , 6×6 , 8×8 , 10×10 will provide enough precision in numerical calculations.

Numerical examples were performed on a VAX-11/780 minicomputer. The proposed method can be used to solve the vibration eigenvalue problems of plates. In the first example,

Table 1. Frequencies of simply-supported plate

<i>m, n</i>	Meshes	M.S.E.M.	Warburton (1954)
$\omega_{1,1}$	8 × 8	19.73381	19.74
$\omega_{2,1}$	8 × 8	49.34739	49.35
$\omega_{3,1}$	8 × 8	78.95469	78.95
$\omega_{2,2}$	8 × 8	98.70766	98.64

α^*

$$\omega = \alpha^*(1/L^2)(D/\bar{m})^{1/2}$$

$$\mu = 0.3$$

M.S.E.M.—Multivariable Spline Element Method.

Table 2. Frequencies of plate with simply-supported edges for different meshes

Meshes				
4 × 4	19.73923	49.38513	79.01062	100.75174
6 × 6	19.73886	49.34992	78.95966	98.79032
8 × 8	19.73381	49.34739	78.95649	98.70766
*	19.74	49.35	78.95	98.64

* Warburton (1954).

Table 3. Frequencies of plate with fixed edges for different meshes

Meshes				
4 × 4	36.01731	73.74091	108.82559	134.59386
6 × 6	35.99366	73.43199	108.32896	132.00426
8 × 8	35.98793	73.40506	108.25762	131.64781
*	35.99	73.41	108.3	131.6

* Blevins (1979).

Table 4. Frequencies of plate with two opposite edges simply supported and other edges fixed for different meshes

Meshes						
4 × 4	28.95979	54.86541	69.59309	94.90040	104.38136	131.90736
6 × 6	28.95244	54.76415	69.33943	94.62939	102.36386	129.46341
8 × 8	28.95030	54.74785	69.32851	94.59936	102.24642	129.13942
*	28.95	54.74	69.32	94.59	102.2	

* Blevins (1979).

Table 5. Frequencies of plate with one edge fixed and other three edges simply supported for different meshes

Present method	Meshes					
	4 × 4	23.65015	51.74423	58.74947	86.28145	102.36620
	6 × 6	23.64658	51.68347	58.65156	85.15998	100.38531
	8 × 8	23.64513	51.67347	58.64604	86.14058	100.28543
	*	23.64	51.67	58.65	86.13	100.3

* Blevins (1979).

simply supported with four edges is a square plate; the second example is also a square plate with four fixed edges; the third and fourth examples are square plates with two opposite edges simply supported and the other edges fixed.

The numerical results of the vibration analysis for the plates are shown in the tables. In Table 1, the vibration frequencies are shown for a square plate with simply-supported edges. In Table 2, the vibration frequencies for the meshes are shown. The results rapidly converge to the exact solution. The lowest frequencies of different supported plates are shown in Table 6.

Table 6. Lowest frequencies of thin plate

Plate	Meshes	M.S.E.M.	Warburton (1954)
SSSS	4 × 4	19.73923	19.74
	6 × 6	19.73886	
	8 × 8	19.73381	
FFFF	4 × 4	36.01731	35.99
	6 × 6	35.99366	
	8 × 8	35.98793	
SFSF	4 × 4	28.95979	28.95
	6 × 6	28.95244	
	8 × 8	28.95030	
SSSF	4 × 4	23.65015	23.65†
	6 × 6	23.64658	
	8 × 8	23.64513	

S—Simply-supported edge.

F—Fixed edge.

† R. D. Blevins (1979).

CONCLUSIONS

Bicubic spline functions are used to construct moments, twisting moments and displacements as field variables for the analysis of plate vibrations in this paper. Multivariate spline eigenvalue equations are derived based on the Hellinger-Reissner mixed energy principle. The present results demonstrate the good convergence characteristics of the multivariable spline element method. The spline functions also have the desired properties associated with piecewise polynomials so that the present method has a high degree of accuracy for the vibration analysis of plates.

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